Notes on Powers and Indices

Positive Integer Indices:

1. Index notation looks like this: x^n and is read as 'x to the power of n'

   x is called the **base** and it is a number that is to be multiplied by itself. It can take any value.

   n is the **index** (plural indices) and tells you how many times the base is multiplied by itself

   For example, $2^4 = 2 \times 2 \times 2 \times 2$ and its value is 16.
2. There are special names for x^2 and x^3: They are 'x squared' and 'x cubed'.
3. If you **multiply** two powers with the **same base** together, you **add their indices**: $x^n \times x^n = x^{m+n}$

   For example, $3^2 \times 3^3 = 3^5$. This is the addition rule of indices, used to multiply powers.
4. If you **divide** two powers with the **same base**, you **subtract their indices**: $x^n \div x^n = x^{m-n}$

   For example, $4^5 \div 4^3 = 4^2$. This is the subtraction rule of indices, used to divide powers.
5. $\boldsymbol{x^1 = x}$

Other Values of Indices

It is clear what a power means when it has a positive integer index, but mathematicians are fond of asking "What if?" Suppose an index were zero, or negative, or a fraction, or even a real number. The answers to these questions are not obvious. Here they are:

6. $\boldsymbol{x^0 = 1}$. Anything to the power of zero is one.

   It is tempting to jump to a wrong answer: 0. No, it is NOT 0, it is 1.

   Why? It follows from the addition law: $x^n + x^0 = x^{n+0} = x^n$.

   So multiplying by x^0 leaves the original power unchanged.

   The only number that leaves another unchanged when multiplying is 1.
7. A **negative index** gives the inverse of the positive power: $\boldsymbol{x^{-n} = \frac{1}{x^n}}$. (Note: It is NOT $-x^n$.)

   Why? Consider this: $x^{-n} \times x^n = x^{-n+n} = x^0 = 1$

   If $x^{-n} \times x^n = 1$, then dividing both sides of the equation by x^n gives $\frac{x^{-n} \times x^n}{x^n} = \frac{1}{x^n}$

   The x^n cancels out on the left hand side, giving $x^{-n} = \frac{1}{x^n}$
8. A **fraction index** seems even more weird at first.

   Consider $x^{\frac{1}{2}} \times x^{\frac{1}{2}}$. By the addition rule, $x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^1 = x$. So, $x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x$.

   The only way to make sense of this is to rewrite it as $\sqrt{x} \times \sqrt{x} = x$.

   So, $x^{\frac{1}{2}} = \sqrt{x}$.

   By a similar argument, $x^{\frac{1}{3}} \times x^{\frac{1}{3}} \times x^{\frac{1}{3}} = x$ and so, $x^{\frac{1}{3}} = \sqrt[3]{x}$. And $x^{\frac{1}{4}} = \sqrt[4]{x}$, and so on.

   So, $\boldsymbol{x^{\frac{1}{n}} = \sqrt[n]{x}}$

   That's OK for indices with a numerator of 1, but what if it is some other value, for instance $\frac{2}{3}$?

   Consider $x^{\frac{2}{3}} \times x^{\frac{2}{3}} \times x^{\frac{2}{3}}$. By the addition rule, $x^{\frac{2}{3}} \times x^{\frac{2}{3}} \times x^{\frac{2}{3}} = x^2$. So, $x^{\frac{2}{3}}$ can only be $\sqrt[3]{x^2}$. Another way of looking at this is shown in point 10.
9. Raising **a power to a power**: What does $(x^n)^p$ mean?

   Consider a simple integer example: $(2^3)^2 = (2 \times 2 \times 2)^2 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6$. As this example shows, the indices are just multiplied. So, $\boldsymbol{(x^n)^p = x^{np}}$.
10. Going back to fractional indices, we saw in point 8 that $x^{\frac{2}{3}} = \sqrt[3]{x^2}$. Point 9 shows that $x^{\frac{2}{3}} = \left(x^{\frac{1}{3}}\right)^2$.

    So, in general you can treat fractional indices as $\boldsymbol{x^{\frac{n}{p}} = \sqrt[p]{x^n}}$ or as $\boldsymbol{x^{\frac{n}{p}} = \left(\sqrt[p]{x}\right)^n}$. Either is correct.